MATHEMATICAL PLAY: ACROSS AGES, CONTEXT, AND CONTENT

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Mathematical play has a fairly short history, with strong roots further back in time (e.g., Papert, Montessori), and understanding the role of mathematical play from early childhood to adulthood is, as yet, unmapped. This working group will build on the success of last year's working group and continue to provide a community space to explore and discuss mathematical play broadly, ranging from informal to formal contexts and from 3rd grade students to teachers in professional development. We will emphasize physical and digital interactions designed specifically to support mathematical play. Each day will focus specifically on a different approach to and definition of mathematical play. Day 1 will focus on mathematical play and making by 3rd to 5th graders across formal and informal environments; Day 2 will focus on intellectual play within mathematical microworlds; and Day 3 will focus on mathematical play in teacher professional development. Throughout the sessions, we will be examining threads of common ground that will assist in developing a more flexible and appropriate model of mathematical play that can inform design of environments and activities across age groups, content, and context.

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Mathematical play has a fairly short history, with strong roots further back in time (e.g., Papert, Montessori). The majority of research on this topic stems from early childhood research on play and researchers have begun to identify the mathematical play children naturally engage in during open-ended play activities and explores how to further mathematize that play and the consequent learning (see Wager & Parks, 2014, for a review). In addition, researchers have explored how mathematicians in the course of their work engage in mathematical play (e.g., Holton, Ahmed, Williams, & Hill, 2001). Given that young children and mathematicians both engage naturally in mathematical play, there is an intriguingly underexplored area of promise between those two populations. A small number of researchers have examined how to support students in approaching mathematics problems with a playful bent (e.g., Steffe & Wiegel, 1994; Holton et al., 2001), but understanding the role of mathematical play from early childhood to adulthood is, as yet, unmapped.

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Following the success of our initial working group at PME-NA 2018, we reached out to new collaborators who will be conducting the activities this year. Our goal is to continue providing a community space to explore and discuss mathematical play broadly, which requires that we extend into new mathematical play activities. In 2018, we had a different focus each day: early childhood mathematical play with wooden blocks (Reimer); middle-school mathematical play with touchpad games (Williams-Pierce); and undergraduate mathematical play with Rubik's cubes (Plaxco). This year, our three foci are: mathematical play in informal makerspaces (Simpson); intellectual playgrounds (Sinclair and Guyevskey); and mathematical play professional development with teachers (Burke and Orrill).

Our goal for the second instantiation of this working group is to continue facilitating mathematical play experiences and discussions around open research questions that transcend our individual lines of research, such as:

- What is the nature of mathematical play across the age/grade bands?
- What are the features, characteristics, and affordances of mathematical play?
- How might context (e.g., physical, digital) influence mathematical play?
- How might content (e.g., fractions, group theory) influence mathematical play?
- How might factors such as gender, race and ethnicity, and parental income/education level influence experience of and access to mathematical play?
- How are mathematical play and mathematical learning related?
- How does mathematical play influence problem solving?
- How do mathematicians (experts) engage in mathematical play, and how might that mindset be fostered for learners (novices)?
- When might a didactical introduction to the content support more productive mathematical play?
- How might mathematical play support or influence learning in other disciplines (e.g., a broader STEM perspective)?

Although answering all of these is beyond the scope of possibility for our working group, we will use these questions to facilitate and orient discussions during each of the three days, and as potential topics for future collaborative investigations. In order to ground the discussions, we will facilitate a mathematical play experience each day, then guide the discussion towards the mathematics at play (pun intended) and the specific characteristics of that mathematical play. During these group discussions, we will regularly orient the conversation specifically towards the open research questions listed above. We will take notes during these conversations and conclude each session by collecting names and emails of working group attendees.

Definitions of Mathematical Play

There are numerous definitions of mathematical play, each emerging from different contexts and with students of different ages. Contexts can range widely, such as digital (Steffe & Wiegel, 1994; Williams-Pierce, 2016, 2017; Sinclair & Guyevskey, 2018), physical (Sarama & Clements, 2009; Simpson, this submission), or paper and pencil-based (Holton et al., 2001). One of the crucial open questions that we highlighted in our discussions at the first working group that we plan to continue discussing is how the definitions may vary in features they prioritize due to the differing contexts. For example, might Holton et al. (2001) and Williams-Pierce (2016) find common ground if they examined similar contexts? Or are their approaches too fundamentally different to ever come to agreement? Or how might Reimer's wooden block activities with young children relate to Plaxco's Rubik's cubes work with undergraduates learning about groups? How might all these examples of play compare to what mathematicians do? We strive to ensure that each session of the working group is oriented around a specific definition and operationalization of mathematical play, so that attendees have concrete experiences grounded in different definitions to facilitate discussion across these different frameworks.

In the below sections - Retrospective (about our initial working group in 2018) and Working Group Schedule and Activities (where we outline the plan for this year's working group) - we describe in more detail our definitions of mathematical play along with activities they inspired.

The History of This Working Group

As mentioned above, this proposal is for a second year of the mathematical play working group, following the success of our first at PME-NA 2018 (Williams-Pierce, Plaxo, Reimer, Ellis, & Dogan, 2018), where we had about 30 attendees each day - most returning each day. We followed the same approximate format for each session: Frame - a brief introduction to the theoretical grounding of that day's topic; Play - a mathematical play activity in order to ground the discussion in a common experience; and Discuss - a broad discussion of the activity, the theoretical underpinnings, and implications for designing and understanding mathematical play and learning. Importantly, we had found in previous working groups (e.g., Nathan, Williams-Pierce, et al., 2017), that beginning with a relevant activity quickly develops fruitful discussions between participants - especially given that working groups tend to have a variety of attendees with wide-ranging expertise in the topic - so we considered the Play component of our format to be crucial for grounding the discussion in shared experiences. One of the crucial goals for our working group was - and is - to ensure that every attendee actually *experiences* mathematical play, and we were delighted that during the 2018 sessions, considerable laughter emerged from our assigned room. However, it is important that we also emphasize the richness of the discussions about mathematical play and learning that emerged alongside the laughter.

The first author was bemused by some of her conversations with PME-NA attendees who did not attend the working group, who often asked, "Are you just playing in the sessions?" An implicit attendant to this question is, "Are people *learning* anything in your working group?" In our retrospectives below, we describe the activities and the discussions that took place, but we would like to take this opportunity to answer this question head-on. First, in order to understand and discuss the role of mathematical play in mathematical learning, we must *experience* what mathematical play feels like. Second, we posit that many PME-NA attendees regularly enjoy their experiences with mathematics, and consequently often engage in some type of mathematical play already in their daily professional lives. Then, discussions about mathematical play can help attendees develop language in order to explicate their own experience with mathematics, and consider new ways in which to introduce elements of mathematical play in their own work with learners. Third, discussion and collaborations that emerged from the relationships established in the working group give empirical evidence that attendees were already considering how play might be infused into mathematical learning - we know of at least four different partnerships that have emerged as a direct consequence of this working group.

Here, we give succinct descriptions of the activities and discussions that occurred in each session last year.

Retrospective - Day 1 (Reimer)

Frame. The first session began with a brief background on early childhood play that highlighted the spontaneous and emergent nature of early play. We focused this session around early childhood play frameworks that emphasize child agency through spontaneous interaction, choice, and opportunities for repeated trials (Wager & Parks, 2014). Because attendees would be engaging in exploratory play in solo and collaborative forms, we also drew on conceptualizations of play that suggest players use novel ways to generate norms, players create new rules in contextual ways to continue play, and play continues through the creation of new meaningful constraints (Di Paolo, Rohde, & De Jaegher, 2010).

Play. We distributed colorful nontraditional pattern blocks (e.g., a mix of both concave and convex hexagons) to participants and encouraged them to begin with individual construction or puzzle play to explore the characteristics of the blocks. We asked them to pay attention to any material constraints that contributed to the ways they developed norms in their individual play. Then we asked attendees to orient each other to their play by explaining their norms and sharing their constructions. This led to opportunities for negotiated play in which participants joined each other in coordinated constructions. Finally, we asked participants to form small groups and bring their shapes together into one as an example of an intertwined sense-making activity. Participants circulated the room and shared the ways their groups had coordinated their play, pointing to specific aspects of their constructions that were made possible by constraints, unexpected possibilities, or the breaking of their established rules.

Figure 1: Attendees Orient Peers to Their Play (left) and Negotiate Coordinated Play (right)

Discuss. We offered several questions to guide participants' discussions after their play experiences, including what mathematics learning opportunities emerge in children's play, and how can teachers support children's mathematics learning without disrupting play? Discussions around these questions centered on the role mathematical properties of the blocks played as constraints in construction play. One aspect of these blocks is that they are stable enough to allow construction in an upwards direction (such as in Figure 1), which a small number of participants noticed and then spread to other participants in the room. Participants also noted productive tensions between self-direction in mathematical play and the ways norms and rules of play are emergent and negotiated. For example, one group had a long discussion about their differing views of using symmetry as a design tool - most group members were striving specifically for a fully symmetric shape, while another group member went so far as to (playfully) hide a piece in order to prevent symmetry from being achieved.

Retrospective - Day 2 (Williams-Pierce)

Frame. This session began by defining mathematical play as *voluntary engagement in cycles of mathematical hypotheses with occurrences of failure*, and introducing five features of digital contexts that support mathematical play: (1) consistent and useful feedback; (2) high enough levels of difficulty and ambiguity that players experience frequent failure that is closely paired with the feedback; (3) non-standard mathematical representations and interactions; (4) mathematical notation introduced late or not at all; and (5) the legitimate possibility of alternative conceptual paths for successful progression (Williams-Pierce, 2017; Williams-Pierce & Thevenow-Harrison, in revision). This framework emerges from a blend of scholarship on mathematical learning and videogames research, the latter of which has embraced the conceptualization of *failure* as an important and often enjoyable experience within the realm of gameplay (e.g., Juul, 2009; Litts & Ramirez, 2014), which highlights the importance of a non-threatening environment where mistakes are perceived as natural and appropriate.

Play. We paired attendees and distributed iPads with *Dragonbox 12+* by WeWantToKnow (a commercial learning game that focuses on balancing equations) and a variety of other mathematics learning games. We chose *Dragonbox 12+* for three reasons: it instantiates four of the five features above, and at least partially instantiates the remaining feature, #2; it is highly popular (over a million downloads worldwide, and a *Wired* article that touted the original game's release; Liu, 2012); and because two of the authors (Williams-Pierce and Ellis) had strongly divergent views about how mathematical the play in *Dragonbox 12+* actually is. Figure 2 shows a screenshot of *Dragonbox 12+* and attendees at the working group playing the game together.

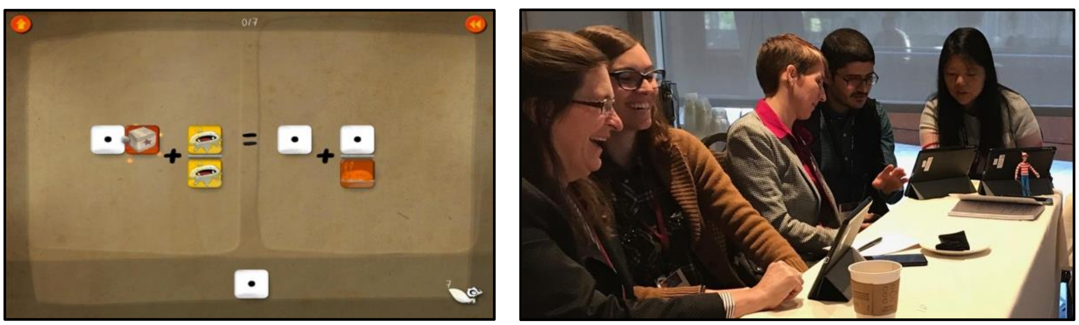


Figure 2: *Dragonbox 12+* (left) and Working Group Attendees Playing Together (right)

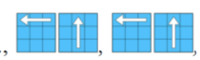
Discuss. To support attendees in analyzing their play experiences, we developed broad guiding questions: (1) Where's the math? (2) Where's the learning? (3) Where's the play? Did you experience *voluntary engagement in cycles of mathematical hypotheses with occurrences of failure*? (4) Does the game have all five features for supporting mathematical play? Paired players discussed the game and these questions as they played *Dragonbox 12+* and the discussions grew naturally into larger groups until we were all debating together as a single large group. The primary group discussion focused on the idea of target content - that is, we discussed how the game designers' intended outcomes may differ from the intended content goals of the individual instructor. One insight that developed was that we need to reframe our discussions about mathematics games from "What math games are good?" to "What games are good for *x*?" where *x* represents a specific learning goal or mathematical behavior. For example, while it may be easy to dismiss many mathematics games as digital versions of flash cards, some instructors may have learning goals that involve practicing math facts - and consequently, in that context, digital flashcards can be a good game. We also discussed how the intended learning goals of designers may not be accomplished - for example, while *Dragonbox 12+* was enjoyable for most attendees, there was debate as to whether the game actually teaches the balancing of equations as intended. To that end, we discussed ways in which games that use innovative interactions and representations can be bridged into more formal and rigorous mathematical learning.

Retrospective - Day 3 (Plaxco)

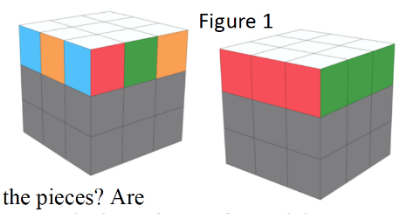
Frame. The Rubik's cube session began with an introduction to the history of the cube, a discussion of some rules governing the cube's movement and arrangements, and an overview of how to communicate moves on the cube (for examples, see Figures 3 and 4). Because research in this area of mathematical play is less developed and we anticipated our participants' unfamiliarity with solving the cube, we focused on first engaging the participants in play intended to support later discussion of the possibilities for educational research.

This play was more structured than the previous days' free play. Given the nature of Rubik's cubes, and how quickly new players can lose their place (mix up the cube), we opted to scaffold working group participants in their interactions by providing simple goals that still supported open-ended playfulness and also provided safety in "failure" to keep the cube solved. The purpose of this variety of structured play was to move participants beyond the typical singular (and often daunting) goal of solving the Rubik's cube in order provide them with alternative goals that would begin and end with the solved cube and other goals that participants could work toward from an unsolved cube. This supported a discussion of how using artifacts of play in alternative ways can open opportunities for creativity and improvisation in play.

Match Game '18: Create an algorithm in secret, record the moves you made to complete it, and show your partner the resulting cube. See if they can match it and "decode" your algorithm (*Tip: start small, with 2 or 3 moves*).

How Many Times: Make up a short algorithm and repeat it (e.g.,  …) until the cube returns to the solved state. Record the algorithm and how many times you completed the sequence. This is called the algorithm's "order." This can get out of hand pretty easily, so be prepared to go for a while.

Here are some games you can play with a mixed-up cube:



Solid Side: Try to complete one side of the cube (Figure 1). For an extra challenge, make all of the pieces along each of that side's edges match each other as well (Figure 1, right versus left).

Rules of the Cube: Try to write out the rules of the cube. Nothing is too obvious. Are there any dependence relations between any of the pieces? Are there any impossible configurations? What makes it so difficult? Try to articulate what you're noticing.

The Pattern Game: Ask an organizer for a pattern card.

Figure 3: Structured Play Beginning and Ending with a Solved Rubik's Cube

Play. We distributed 50 solved 3x3x3 Rubik's cubes to the working group participants, who engaged in a number of types of structured play. We guided this play with use of a handout that provided a list of games that begin (and hopefully end) with a solved cube (Figure 3, top) as well as some that drew on the likelihood of the participants' eventual mixing up of the cube during the first type of organized play (Figure 3, bottom). Throughout this time, the organizers of the working group walked around to engage with participants as they explored the Rubik's cubes. Very fortunately, a few of the participants happened to have experience solving Rubik's cubes, which allowed for some "safety" in the other participants' *failure* to keep the cube near a solved state. This occurred when participants "lost track" of the moves that they needed to make relative to the moves they had already made. Initially, we had planned to incorporate iPads in the activity so that partners could record each other to help keep track of moves. However, we decided that we could forego this by incorporating tasks and games that could be completed on unresolved cubes. Specifically, we included a "Pattern Game" that tasked participants with matching a face of the cube to an image on a small, numbered card. These cubes were then collected with the cards and used to construct a mosaic (Figure 3, bottom) that read **PME NA 2018**!

Figure 4: The Last Remaining Working Group Participants with a Rubik's Cube Mosaic

Discuss. During the discussion portion of the third day, the organizers brought up a few themes that had continued to emerge throughout the sessions. For instance, the group tried to elaborate on what aspects of the Rubik's cube made play with it mathematical. This included a distinction between an intended content focus of mathematical play and a practice focus of mathematical play. For instance, the Rubik's cube exemplifies the mathematical structure of an algebraic group. However, for many K-12 students who are interested in cubing, the formal construct of a group is unnecessary to engage in and develop their own mathematical practices of problem solving, communicating, generalizing, and reasoning structurally that play with the Rubik's cube can afford. This conversation supported a more general discussion about the values that educators have when focusing on students' mathematical play, specifically the need for an awareness of our focus on content and practices as we develop mathematical play spaces.

Working Group Schedule and Activities

Each day will focus specifically on a different approach to and definition of mathematical play. Day 1 will focus on mathematical play and making by 3rd to 5th graders across formal and informal environments (Simpson); Day 2 will focus on intellectual play within mathematical microworlds (Sinclair and Guyevskey); and Day 3 will focus on mathematical play in teacher professional development (Burke). While each session has distinct differences in their definitions of mathematical play and approaches to fostering such play, there are commonalities across the sessions. For example, the art produced by the Art Bots on Day 1 have distinct similarities to the mathematical microworld focused on in Day 2. Day 2 and Day 3 both highlight the use of open-ended microworlds for supporting mathematical play, although the microworlds focus on different types of content (geometry versus ratio and proportion). By concluding the working group with a session on teacher professional development, we hope to foster conversations about how to support both others and ourselves in fostering mathematical play experiences.

Day 1 Session – Following Williams-Pierce (2016)

The first session will focus on Dr. Simpson's preliminary research on mathematical play of 3-5 graders in making and tinkering contexts across formal and informal learning environments. As such, building upon the scholarship of Williams-Pierce (2016), mathematical play is defined as *voluntary engagement in cycles of mathematical hypotheses, wonders, and curiosities that lead to or stem from occurrences of failure*. Making and tinkering is not a new phenomenon but involves youth in the process of designing, constructing, testing, and revising of physical and/or digital products for play or for purpose. It involves the use of a variety of materials and tools such as low-tech (e.g., conductive tape), high-tech (e.g., 3D printers), household items (e.g., cotton balls), and recyclable material (e. g., yogurt containers). Scholarship on making and tinkering contexts has illustrated youths' engagement in "experimental" play as scientists and engineers (e.g., Simpson, Burris, & Maltese, 2017), but less is known regarding youths' play as mathematicians in such contexts (Pattison, Ruben, & Wright, 2016).

The session will begin by engaging participants in a making activity – participants will make an Art Bot that draws a hands-free art piece (see Figure 5). During and after this activity, we will guide discussions about how (and if) participants felt their experience involves mathematical play. For example, was mathematical play a genuine part of their making experience or did it seem like an add-on or an outlier to their experience? Did mathematizing happen during play, or only afterwards, when we engage in overt discussions about mathematical play? If the latter, can we trace the mathematics brought up afterwards back to the play, to see how the play provoked our own mathematizing and consequent play? We will particularly focus on the role of failure in their experiences of mathematical play, given Simpson's previous work examining failure episodes in makerspaces (e.g., Simpson & Maltese, 2017) and the emphasis on failure in Williams-Pierce's (2016) definition of mathematical play.

Figure 5: Example of an Art Bot (left), and the Resulting Art Piece (right)

Next, Dr. Simpson and Dr. Williams-Pierce will present episodes of how one youth's mathematical play in making and tinkering context span space (e.g., after-school program vs. home) and time. This is a dynamical view of youths' play, as opposed to a "flat" view, as we consider how mathematical play and the physical and abstract objects transcend time (time scales; Lemke, 2000) and environment (boundary crossing; Akkerman & Bakker, 2011). We will conclude this session by seeking feedback on our view of the ways in which mathematical play and learning can transcend contexts, and discussing future research possibilities.

Day 2 Session – Following Featherstone (2000)

The second session, conducted by Dr. Sinclair and Ms. Guyevsky, will focus on Featherstone's (2000) *intellectual play*. Drawing on the work of Huizinga, Helen Featherstone (2000) argues for the central importance of what she calls "intellectual play" in mathematics learning. She finds many parallels between the characteristic features of play and the way mathematicians work. After having observed her students in an elementary classroom engaged in serious mathematical activity in a manner she described as "math in the ludic zone", Featherstone wondered whether we could help children see mathematics as an arena for play. She is not arguing that play and mathematics are identical, nor that teachers should include play periods as part of their mathematical lessons; instead she is interested in "the moments in which doing mathematics becomes playful and about the ways in which play might expose children to aspects of the discipline that may not ordinarily be visible to them" (p. 16).We conjecture that certain well-designed and open-ended computer-based environments might be especially effective at providing "playgrounds" or mathematical microworlds in which play can occur. As participants engage in playful activities, they will be invited to reflect on how such play might free learners "from the dictatorship of concrete objects," as Vygotsky (1933/1966) writes, and enable them to "develop the capacity to behave in accordance with meaning" (p.19).

The Web Sketchpad version of *The Geometer's Sketchpad* (Jackiw, 2012) will provide the digital context for intellectual play. In contrast with traditional definition of a game, the participants will not be offered a set of explicitly prescribed rules that the player has to abide by; rather, the rules of the "game" will be determined by the fixed geometric properties of the figures and communicated to the player via visual feedback as participants engage with the microworld. The player "wins", if the outcome of the interaction is a robust dynamic construction that behaves according to the criteria of the assigned task. However, being "at play" is more of our focus here than "winning", as we hypothesize that those participants who "lost" the game have nonetheless come a step closer to understanding a certain geometric concept. We chose a dynamic geometry environment because it offers various modes of feedback that can enable students to experiment, test conjectures and debug without having to appeal to an outside authority, which we think is a crucial aspect of enabling play (Sinclair & Guyevskey, 2018).

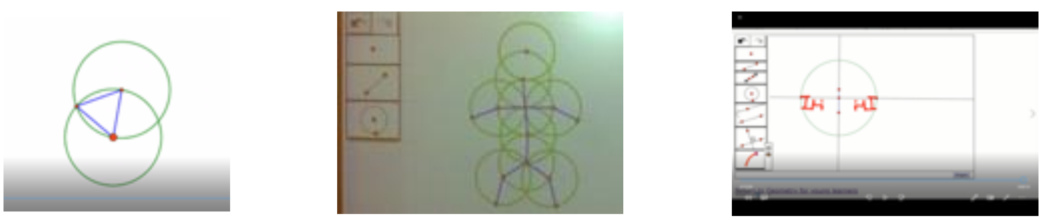


Figure 6: Regular Polygons (left), Stickman (middle), and Mirror Machine (right).

Participants will be asked to engage in computer-based mathematical tasks in which they will explore geometric concepts and relations, and model mathematics in contextualized experiences. Activities will be carried out in pairs. Participants will be offered three different tablet-based activities that are potentially relevant to play. These activities will vary in the level of difficulty, i.e., the geometric relationships involved in a task will range from less to more complex. A progression of tasks will focus on regular polygons, properties of a circle, and the concept of symmetry. In Task 1, Regular Polygons, participants will construct a variety of shapes, beginning with an equilateral triangle, and then gradually tessellating with that triangle to construct a rhombus, regular trapezoid, and regular hexagon (Figure 6, left). In Task 2, Stickman, participants will be asked to construct a stickman in such a way, that the arms are equilateral, and the legs are equilateral. As extension, equilateral fingers could be added, as well as neck, knees and elbows. Participants will need to accommodate for the stickman's ability to grow limbs while maintaining their congruency (Figure 6, middle). In Task 3, Mirror Machine, participants will be shown a picture of Leonardo da Vinci mirror-writing machine that he designed to encode his writings, and then shown a dynamic version of such a machine in web sketchpad. They will be invited to create such a machine themselves (Figure 6, right).

Day 3 Session – Following Burke (2017)

The third and final session, conducted by Dr. Burke and Dr. Orrill, will focus on mathematical play in teacher professional development. Dr. Burke will coordinate an exploration of the environments and tasks of Dr. Orrill's Proportions Playground project. This approach to professional development relies on engagement with digital "toys" designed by Dr. Burke and Dr. Orrill to encourage participants to "play" with mathematics to strengthen their understanding of important aspects of ratio and proportion. The toys we'll be playing with in the workshop were designed to be explored, requiring us to conjecture, justify, and explain as we collaborate. We will focus on using the Bars Toy, which is built around a relatively simple browser-based simulation of two bars of interdependent length that can be edited by dragging. In the PD and in the working group session, we will use three separate scenarios to provide contrasting mathematical relationships that will elicit discussions in which mathematical language and ideas are valuable to differentiate among them.

For our purposes, we define playing with math to mean engaging teachers in problem solving in a way that relies on making and testing conjectures and mathematical arguments that can be reasoned about, tested, illustrated, and explained through the use of dynamic tools. In order for this play to occur, an environment (including the toys and the implementation of the PD) has to allow for playfulness. It has to exist as a space where participation - and consequently play - is a safe activity. We use language and norms to lower the stakes of participation, and provide activities that teachers can immediately explore and use to form conjectures (Burke, 2017).

Central to the goal of our PD were three key ideas of proportionality: quantity, constant, and covariation. When we talk about proportional situations, what quantities do we refer to? What do we identify as remaining constant? How do we, and do we, use covariation to make sense of proportion? As Sutherland and Balacheff (1999) have written, students must learn to use the language of algebraic thinking in order to develop algebraic ways of solving problems. While teachers have shown themselves to be adept at cross-multiplying missing value problems, we want them to be able to see a ratio as a comparison between two quantities (Lamon, 2007). In particular, we want them to be able to talk to students about proportional situations as ones in which "the ratio of one quantity to the other is invariant as the numerical value of both quantities change by the same scale factor" (Lobato & Ellis 2011, p. 11). To this end, we developed a PD in which rich discussions about proportional situations can take place, allowing teachers to use their knowledge of proportion together with the key ideas we emphasize.

In this session, after facilitating a mathematical play experience with the Bars Toy, we will highlight some of our unexpected findings or struggles from our PD research, and foster a discussion that feeds into our re-design plans. In particular, we plan to use this opportunity to both share our work on mathematical play with teachers, and to gain insight from a community of deep thinkers about mathematical play in order to further our ongoing PD design. Our broad discussion questions will revolve around the relationship between mathematical play and learning for teachers, and how can support teachers in bridging from their own experiences of mathematical play in the PD to fostering such play with their students.

After Dr. Burke's group activity, we will conclude the working group by discussing our experiences across each of the three days. In particular, we will focus on the three mathematical play approaches emphasized on different days, and how the differing contexts and content may have influenced the development of these approaches, following the open research questions identified above. We will seek to find threads of common ground that will assist in developing a more flexible and appropriate model of mathematical play that can inform design of such environments and activities across age groups, content, and tools. Finally, we will conclude by identifying next steps for the working group members, as outlined in the following section.

Future Plans

In order to establish clear next steps at the working group, we will investigate potential NSF conference proposals in advance, and share a draft submission action plan with attendees, with the goal of collaboratively conducting a workshop that draws across different areas of mathematical play expertise, in order to begin productively synthesizing the phenomenon.

In addition, this working group will continue establishing a network of support for designing and examining mathematical play at all ages. To that end, we will continue curating the listserv we established after last year's session, so that attendees find it convenient and simple to continue the pattern of fruitful collaborations that the first session spawned.

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