Mathematical Play: Across Ages, Context, and Content

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MATHEMATICAL PLAY: ACROSS AGES, CONTEXT, AND CONTENT

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Mathematical play has a fairly short history, with strong roots further back in time (e.g., Papert, Montessori), and understanding the role of mathematical play from early childhood to adulthood is, as yet, unmapped. This working group will provide a community space to explore and discuss mathematical play broadly, ranging from early childhood to undergraduate learners, with an emphasis on physical and digital interactions designed specifically to support mathematical play. Each day will focus specifically on a different approach to and definition of mathematical play. Day 1 will focus on early childhood mathematical play with pattern blocks; Day 2 will focus on mathematical play in videogames; and Day 3 will focus on using Rubik's Cubes to investigate undergraduate mathematics, namely algebraic groups. Throughout the sessions, we will be examining threads of common ground that will assist in developing a more flexible and appropriate model of mathematical play that can inform design of environments and activities across age groups, content, and context.

Keywords: Instructional activities and practices; Design experiments; Technology

Mathematical play has a fairly short history, with strong roots further back in time (e.g., Papert, Montessori). The majority of research on this topic stems from early childhood research on play, and researchers have begun to map out the phenomenon of mathematical play in young children (see Wager & Parks, 2014, for a review). This approach identifies the mathematical play children naturally engage in during open-ended play activities, and explores how to further mathematize that play and the consequent learning. In addition, researchers have explored how mathematicians in the course of their work engage in mathematical play (e.g., Holton, Ahmed, Williams, & Hill, 2001). Given that young children and mathematicians both engage naturally in mathematical play, there is an intriguingly underexplored area of promise between those two populations. A small number of researchers have examined how to support students in approaching mathematics problems with a playful bent (e.g., Steffe & Wiegel, 1994; Holton et al., 2001), but understanding the role of mathematical play from early childhood to adulthood is, as yet, unmapped.

This working group will provide a community space to explore and discuss mathematical play broadly, ranging from early childhood to undergraduate learners, with an emphasis on physical and digital interactions designed specifically to support mathematical play.

The Origin of this Working Group

The authors have been engaging in researching mathematical play in their own arena of interest for some time and are beginning to engage in conversations across age groups (e.g., early childhood, undergraduate), content (e.g., patterns, fractions, algebraic groups), and contexts (e.g., Rubik's cubes, videogames). This working group proposal is the first collaborative effort from this diverse group of researchers.

Our goal for this working group is to facilitate mathematical play experiences and discussions around open research questions that transcend our individual lines of research, such as:

- What is the nature of mathematical play across the age/grade bands?
- What are the features, characteristics, and affordances of mathematical play?
- How might context (e.g., physical, digital) influence mathematical play?
- How might content (e.g., fractions, group theory) influence mathematical play?
- How might factors such as gender, race and ethnicity, and parental income/education level influence experience of and access to mathematical play?
- How are mathematical play and mathematical learning related?
- How does mathematical play influence problem solving?
- How do mathematicians (experts) engage in mathematical play, and how might that mindset be fostered for learners (novices)?
- When might a didactical introduction to the content support more productive mathematical play?
- How might mathematical play support or influence learning in other disciplines (e.g., a broader STEM perspective)?

Although answering all of these is beyond the scope of possibility for our working group, we will use these questions to facilitate and orient discussions during each of the three days, and as potential topics for future collaborative investigations.

Definitions of Mathematical Play

There are a variety of definitions of mathematical play, each emerging from different contexts and with different ages. In this section, we briefly review a selection of those definitions in order to contextualize the activities we have planned for the working group sessions.

Steffe and Wiegel (1994) conducted a teaching experiment with a digital microworld designed to support operations with quantities, and the role of the instructor was directly to shift students from engaging in independent mathematical activity into engaging in mathematical play. Appropriately, they then defined mathematical play as "independent mathematical activity with a playful orientation" (p. 27). Sarama and Clements (2009) took a different approach: instead of focusing on supporting the shift from independent mathematical activity to mathematical play, they distinguish between *play that involves mathematics* and *mathematical play*. They give examples of the former, including a store in the classroom where one student plays the shopkeeper, and another purchases toy dinosaurs for a dollar apiece, and the students engage in play while also engaging in counting, place value, and practicing arithmetic. For mathematical play, on the other hand, children play with the mathematics itself, and the authors describe a four-year-old who is playing with three of five train engines, and first describes those three train engines as being numbers 1, 2, and 3 – thus missing engines numbered 4 and 5. Then she decides that the trains she has are actually numbered 1, 3, and 5, leaving trains 2 and 4 missing – as Sarama and Clements (2009) note, "she was playing with the idea that counting words themselves could be counted" (p. 327), that is, she was playing directly with mathematics.

Holton et al. (2001) define mathematical play as "that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using the formal rules of mathematics to follow any ideas to some sort of a conclusion" (p. 403). They identify six components of mathematical play:

(1) It is a solver-centred activity with the solver in charge of the process;
(2) It uses the solver's current knowledge;
(3) It develops links between the solver's current schemata while the play is occurring;
(4) It will, via 3, reinforce current knowledge;
(5) It will, via 3, assist future problem solving/mathematical activity as it enhances future access to knowledge;
(6) It is irrespective of age. (p. 404)

They additionally note that mathematical play will not necessarily directly lead to a correct solution of the problem being played with – in fact, one of the important components of mathematical play is that it "provides a non-threatening environment where incorrect solutions are not read as mistakes and may lead to a better understanding of the problem and/or the confrontation of misconceptions" (p. 404).

Another framework for mathematical play (Williams-Pierce, 2016, 2017) also highlights the importance of a non-threatening environment where mistakes are perceived as natural and appropriate. This framework emerges from a blend of scholarship on mathematical learning and videogames research, the latter of which has embraced the conceptualization of *failure* as an important and often enjoyable experience within the realm of gameplay (e.g., Juul, 2009; Litts & Ramirez, 2014). Williams-Pierce (2016, 2017) defined mathematical play as *voluntary engagement in cycles of mathematical hypotheses with occurrences of failure*, and describes five features of digital contexts that support such mathematical play:

(1) consistent and useful feedback;
(2) high enough levels of difficulty and ambiguity that players experience frequent failure that is closely paired with the feedback;
(3) non-standard mathematical representations and interactions;
(4) mathematical notation introduced late or not at all; and
(5) the legitimate possibility of alternative conceptual paths for successful progression.

Each of the definitions of mathematical play described above emerge from different contexts: some digital (Steffe & Wiegel, 1994; Williams-Pierce, 2016), some physical (Sarama & Clements, 2009), some paper and pencil-based (Holton et al., 2001). One of the crucial open questions that we highlighted above is how the definitions may vary in the features that they prioritize, due to the differing contexts. For example, might Holton et al. (2001) and Williams-Pierce (2016) find common ground if they examined similar contexts? Or are their approaches too fundamentally different to ever come to agreement? Each session of the working group is oriented around a specific definition and operationalization of mathematical play, so that attendees have concrete experiences grounded in different definitions to facilitate discussion across these different frameworks.

Working Group Schedule and Activities

For each of the three Working Group sessions, we plan to begin with a hands-on mathematical play activity designed from a different mathematical play approach. The first author has been involved in conducting PME-NA working groups in the past (Nathan et al., 2016, 2017), and found that beginning with a relevant activity quickly develops fruitful discussions between participants. In particular, working groups tend to have a variety of attendees with a wide-ranging level of expertise in the topic, and grounding the discussion in shared experiences serves to support a sense of community. Consequently, we will facilitate a mathematical play experience each day, then guide the discussion towards the mathematics at play (pun intended) and the specific characteristics of that mathematical play. During these group discussions, we will regularly orient the conversation specifically towards the open research questions listed above. We will take notes during these conversations, and conclude each session by collecting names and emails of working group attendees.

Each day will focus specifically on a different approach to and definition of mathematical play. Day 1 will focus on early childhood mathematical play with pattern blocks; Day 2 will focus on mathematical play in videogames; and Day 3 will focus on using Rubik's Cubes to investigate undergraduate mathematics, namely algebraic groups. While each day's session moves up in age (from early childhood to undergraduate), we will also be moving down in terms of established research. In particular, Day 1 showcases the most researched area of mathematical play and will be primarily about introducing attendees to the rich history of early childhood mathematical ply; Day 2 focuses on comparing two instantiations of mathematical play, and seeks to identify similarities and differences across the digital designed contexts; and Day 3 involves the initial design of an undergraduate mathematical play experience, and will focus on gathering input from the attendees on potential areas of redesign.

Day 1 Session – following Wager & Parks (2014)

The first session will focus on mathematical play in early childhood. Mr. Reimer will engage participants in a form of pattern block puzzle play that involves spatial reasoning and the composition and decomposition of shapes. While mathematical play is a generally new area of investigation, Wager and Parks (2014) found that "studies involving block play are the most well documented and longest established" (p. 217), although spatial learning with puzzle play is still under-examined. Furthermore, most geometry and spatial learning studies have taken place in artificial contexts (e.g., research labs), rather than natural contexts. Consequently, this activity relies upon early childhood play frameworks that emphasize child agency through spontaneous interaction, choice, and opportunities for repeated trials (Wager & Parks, 2014), with a focus on exploration with physical objects to support the development of mental transformations.

Mr. Reimer will bring sets of pattern block puzzles for the participants (see Figure 1) – for use in solo activity or small groups, depending upon the number of attendees. He will highlight the importance of spontaneous interaction, choice, and repeated trials, and throughout the session activity, he will provide orienting questions and comments–such as initiated constraints and challenges to extend or deepen the activity–to help participants explore possibilities and mathematize their play.

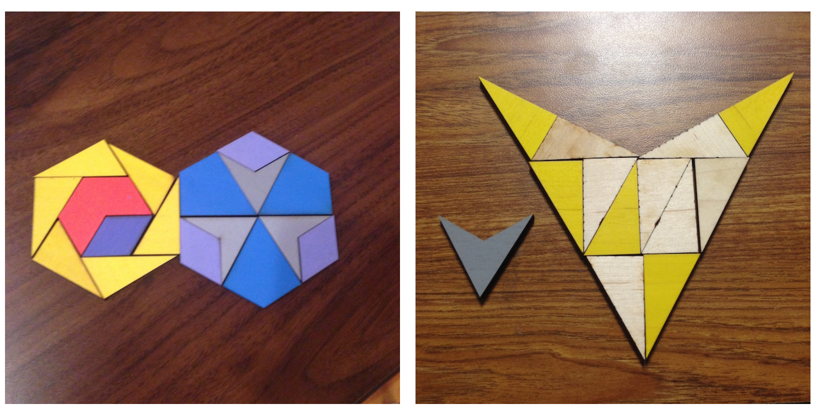

Figure 1. A sample of puzzle designs participants may create.

The activity will conclude with an open discussion that focuses on the affordances of the physical materials in encouraging mathematical play, the importance of orienting questions, and the role of narratives in describing and supporting participants' play activity. Then we will guide the discussion towards the open research questions listed above.

Day 2 Session – following Williams-Pierce (2017)

The second activity will focus on mathematical play in digital contexts. Dr. Williams-Pierce will coordinate this activity by bringing two games that were designed to support mathematical play: *Dragonbox 12+* by WeWantToKnow (a commercial learning game that focuses on balancing equations), and her in-revision game, *Rolly's Adventure* (which focuses on the development of understanding multiplying fractions). Both of these games support mathematical play and learning in distinctly different ways: *Dragonbox 12+* through faithfully enacting mathematical phenomena with novel representations and interactions; and *Rolly's Adventure* through secretly modeling the operation of multiplication and requiring players to engage repeatedly in interpreting feedback through failure in order to unpack that secret operation. See Figure 2 for screenshots of both games.

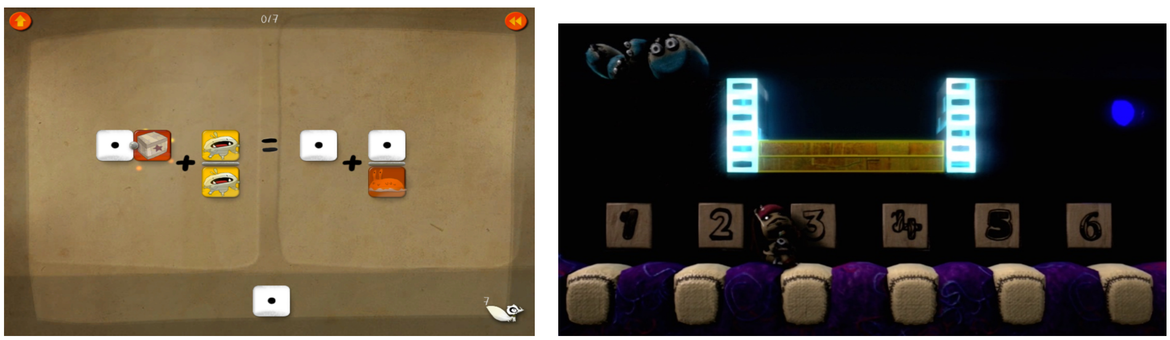

Figure 2. A screenshot each of *Dragonbox 12+* (left) and the original *Rolly's Adventure* (right).

We chose those particular two games for a variety of reasons. First, having the designer of *Rolly's Adventure* present means that she can unpack any of the hidden design decisions behind the game that may emerge as participant questions during play and discussion. Second, while the *Dragonbox* series is highly popular (with over a million downloads worldwide, and a *Wired* article that touted the original game's release; Liu, 2012), co-authors of this working group have strongly divergent views about how mathematical the play in *Dragonbox 12+* actually is. Consequently, we anticipate that these two games will provoke a lively discussion!

Both games are iPad games, and Dr. Williams-Pierce will bring ten iPads to share with participants, as well as instructions for installing her game for free on their own touchpad devices. Depending upon the number of attendees, this activity may be in pairs or small groups. Half the participants will play *Dragonbox 12+*, and half *Rolly's Adventure*, while taking guided notes on their experience of mathematical play with the games. These guided notes will be based upon the five features described above (Williams-Pierce, 2016, 2017). For example, the guided notes will include questions like: how does the game give you feedback on what you are doing well or poorly? How does the experience of *failure* influence how you attend to feedback? What, if any, is the role of mathematical notation? After playing, Dr. Williams-Pierce and Dr. Ellis – as the two authors who disagree about mathematical play in *Dragonbox 12+* – will co-facilitate a discussion about how the games instantiate those five features differently, and how the participants felt their experiences were or were not mathematical play.

Day 3 Session – following Holton et al. (2001)

First, Dr. Plaxco will discuss his preliminary work in designing activities to engage undergraduate students in mathematical play with Rubik's Cubes and connections to the cube's associated algebraic group structure. This will entail an introduction to several algorithms that can be useful for manipulating specific pieces on the cubes as well as a discussion of the usefulness of some group theoretical constructs (e.g. conjugation and commutators) in generating designs on the cubes (see Figure 3). Holton et al. (2001) note that "Mathematical play involves pushing the limits of the situation and following thoughts and ideas wherever they may lead. Hence there are no obvious short-term goals for mathematical play; it is designed to allow complete freedom on the part of the solver to wander over the mathematical landscape" (p. 403). Rubik's Cubes allow for such mathematical play, though players are typically less familiar with the rules and operations of the cubes because the setting is noncommutative, which allows for surprising results when trying to generalize from experiences, and because the "landscape" is rather large – a 3x3x3 cube has over 4.3×10^{17} positions.

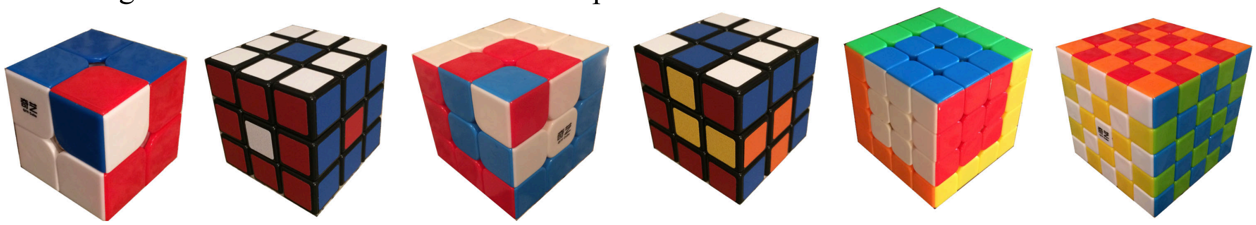

Figure 3. A Sample of Cube Designs That Working Group Participants May Attempt

Dr. Plaxco will distribute a class set of solved 2x2x2 and 3x3x3 Rubik's Cubes so that each member of the working group will have access to a cube. Members will also receive a laminated sheet with instructions for each of the algorithms discussed and a second sheet with possible designs that members can set out to re-create. One difficulty Dr. Plaxco has encountered with introducing Rubik's Cubes to students is the need for prolonged orientation to gain a sense of how to use algorithms to produce desired outcomes. Because of this, we will sequence the tasks so that participants will work to change a solved cube into increasingly difficult designs, incorporating longer algorithms and combining algorithms in new and different ways. Participants will also be invited to explore the possibilities of what they can create with the algorithms and generate their own designs on the cubes. We will give participants iPads to videotape their manipulations of their cubes in order to help them remember the sequences of moves they make. We will ask the participants to try and develop notations to help record these moves as well. This activity will culminate with a "Match Game" in which groups of participants will create designs, recording and notating the required moves and challenge other teams to re-create their designs. We will then close this session with a discussion of potential directions for activities with undergraduate students, including avenues for collaborations in researching mathematical play with the Rubik's Cubes.

After Dr. Plaxco's group activity, in order to conclude the working group, we will discuss our experiences across each of the three days. In particular, we will focus on the three mathematical play approaches emphasized on different days, and how the differing contexts and content may have influenced the development of these approaches, following the open research questions identified at the beginning of the document. We will seek to find threads of common ground that will assist in developing a more flexible and appropriate model of mathematical play that can inform design of such environments and activities across age groups, content, and tools. Finally, we will conclude by identifying next steps for the working group members, as outlined

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in the following section.

Future Plans

This working group is the first step in establishing a network of support for designing and examining mathematical play at all ages. First, we will develop a plan for a white paper about mathematical play that draws out common themes across the differing ages and content that have been researched thus far, and identifies aspects of the frameworks and definitions that may be irreconcilable. For example, might unstructured mathematical play and structured mathematical play be fundamentally different? Are there commonalities about mathematical play that can be established across paper-and-pencil activities, physical manipulatives, and digital contexts? How might audience (e.g., voluntary players who can stop playing at any time, students who have no choice but to participate, research participants who chose to be part of the research) influence the experience of mathematical play? In this paper, we will explore those questions, and identify the particular next steps that should be undertaken to further understand and unpack the phenomenon of mathematical play. Second, we will investigate future potential conference proposals or journal articles that draws across different areas of mathematical play expertise, in order to begin productively synthesizing the phenomenon. For example, one of the discussions that developed during the crafting of this proposal revolved around the use of Holton et al.'s (2001) components of mathematical play for the Rubik's cube activity, and how certain aspects of the Williams-Pierce (2016, 2017) framework might apply better. One potential line of investigation then is to examine the possibility of synthesizing the two frameworks in a way that accounts for the very different contexts they initially emerged from.

Third, we will set up a listserv for the attendees, so that they find it convenient and simple to stay in touch with others interested in mathematical play. Immediately after the conference, we will share this listserv so that participants may continue discussing any potential collaboration plans developed through discussions during the working group. Finally, we plan to submit for an NSF Advancing Informal STEM Learning: Conferences grant in order to conduct yearly meetings that focus solely on mathematical play throughout the lifespan. The participants at the Mathematical Play Working Group at PME-NA 2018 will be the first list we contact to invite to apply for this opportunity.

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